QUANTUM AFFINE ALGEBRAS, CANONICAL BASES AND q-DEFORMATION OF ARITHMETICAL FUNCTIONS

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ABSTRACT. In this paper, we obtain affine analogues of Gindikin-Karpelevich formula and Casselman-Shalika formula as sums over Kashiwara-Lusztig's canonical bases. Suggested by these formulas, we define natural q-deformation of arithmetical functions such as (multi-)partition function and Ramanujan τ-function, and prove various identities among them. In some examples, we recover classical identities by taking limits. We also consider q-deformation of Kostant's function and study certain q-polynomials whose special values are weight multiplicities.

INTRODUCTION

This paper is a continuation of [16]. The classical Gindikin-Karpelevich formula and Casselman-Shalika formula express certain integrals of spherical functions over maximal unipotent subgroups of p-adic groups as products over all positive roots. In [16], we expressed the products over positive roots as sums over Kashiwara-Lusztig's canonical bases ([17, 18]). That idea first appeared in the papers [10, 21] from the context of Weyl group multiple Dirichlet series [8, 9]. (See also [5, 6, 7].) Let G be a split reductive p-adic group, χ be an unramified character of T, the maximal torus, and f^0 be the standard spherical vector corresponding to χ. Let $\mathbf{z}$ be the element of ${}^LT \subset {}^LG$, the L-group of G, corresponding to χ by the Satake isomorphism. Then

$$\int_{N_-(F)} f^0(n)\,dn = \prod_{\alpha\in\Delta^+} \frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha} = \sum_{b\in\mathbf{B}} (1-q^{-1})^{d(\phi_{\mathbf{i}}(b))}\mathbf{z}^{\mathrm{wt}(b)}, \tag{0.1}$$

$$\begin{aligned} \int_{N_-(F)} f^0(n)\psi_\lambda(n)\,dn &= \chi(V(\lambda)) \prod_{\alpha\in\Delta^+}(1-q^{-1}\mathbf{z}^\alpha) \\ &= (-t)^M \mathbf{z}^{2\rho}\,\chi(V(\lambda)) \prod_{\alpha\in\Delta^+}(1-t^{-1}\mathbf{z}^{-\alpha}) \\ &= (-t)^M \mathbf{z}^{\rho} \sum_{b'\otimes b\in\mathfrak{B}_\lambda\otimes\mathfrak{B}_\rho} G_\rho(b;t)\mathbf{z}^{\mathrm{wt}(b'\otimes b)}, \end{aligned} \tag{0.2}$$

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where Δ^+ is the set of positive roots, $\mathbf{B}$ is the canonical basis, $\mathfrak{B}_\lambda$ is the crystal basis with highest weight λ, and we set $M = |\Delta^+|$ and $t = q^{-1}$. Notice that the tensor product of crystal bases behaves well in the Casselman-Shalika formula.

In the affine Kac-Moody groups, A. Braverman, D. Kazhdan and M. Patnaik calculated the integral (0.1) in [4], and obtained a formula of the form

$$\int_{N_-(F)} f^0(n)\,dn = A \prod_{\alpha\in\Delta^+} \left(\frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha}\right)^{\operatorname{mult}\alpha}, \tag{0.3}$$

where A is a certain correction factor. When the underlying finite simple Lie algebra $\mathfrak{g}_{\rm cl}$ is simply-laced of rank n, A is given by

$$\prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{1-q^{-d_i}\mathbf{z}^{j\delta}}{1-q^{-d_i-1}\mathbf{z}^{j\delta}},$$

where d_i's are the exponents of $\mathfrak{g}_{\rm cl}$, and δ is the minimal positive imaginary root.

In this paper, we use the explicit description of the canonical basis due to Beck, Chari, Pressley and Nakajima ([2], [3]) to write the right hand side of (0.3) as a sum over the canonical basis. Moreover, we obtain the generalization of (0.2). Namely, we prove (Theorem 1.16 and Corollary 2.13)

$$\prod_{\alpha\in\Delta^+} \left(\frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha}\right)^{\operatorname{mult}\alpha} = \sum_{b\in\mathbf{B}} (1-q^{-1})^{d(\phi(b))}\mathbf{z}^{\operatorname{wt}(b)}, \tag{0.4}$$

$$\chi(V(\lambda))\mathbf{z}^\rho \prod_{\alpha\in\Delta^+} (1-q^{-1}\mathbf{z}^{-\alpha})^{\operatorname{mult}\alpha} = \sum_{b'\otimes b\in\mathfrak{B}_\lambda\otimes\mathfrak{B}_\rho} G_\rho(b;q)\,\mathbf{z}^{\operatorname{wt}(b'\otimes b)}, \tag{0.5}$$

where $\mathbf{B}$ is the canonical basis of $\mathbf{U}^+$ (the positive part of the quantum affine algebra), and $\mathfrak{B}_\lambda$ is the crystal basis with highest weight λ. Here $\mathbf{z}$ is a formal variable. We also write the correction factor A as a sum over a canonical basis in the case when $\mathfrak{g}_{\rm cl}$ is simply-laced. We first prove (0.4) by induction, and deduce (0.5) from (0.4) and Weyl-Kac character formula. In the course of proof, we see that (0.5) can be considered as a q-deformation of Weyl-Kac character formula.

In the development of Weyl group multiple Dirichlet series ([5, 6, 7, 8, 9]), one of the main problems has been how to define the local coefficient (or p-part). Various combinatorial methods have been adopted to define the coefficient. In particular, the string parametrization (or BZL-path) of a crystal graph was used in [9]. In our previous paper [16], the q-polynomial $H_{\lambda+\rho}(\mu;q)$

was defined using the generating series:

$$\sum_{\mu \in Q_+} H_\lambda(\mu; q)\mathbf{z}^{\lambda-\mu} = \sum_{w\in W} (-1)^{\ell(w)} \sum_{b\in \mathbf{B}} (1-q^{-1})^{d(\phi(b))} \mathbf{z}^{w\lambda - \mathrm{wt}(b)}. \tag{0.6}$$

This definition works for all the finite root systems in a uniform way, and is essentially the same as the local coefficient in the non-metaplectic case that was introduced in the work [12] of Chinta and Gunnells on the construction of Weyl group multiple Dirichlet series.

In this paper, we define the polynomial $H_{\lambda+\rho}(\mu; q) \in \mathbb{Z}[q^{-1}]$ for affine cases using the same formula. That is to say, the definition (0.6) works for affine cases as well. Moreover, we show that the polynomial $H_{\lambda+\rho}(\mu; q)$ has many remarkable, representation-theoretic properties; its constant term is the multiplicity of the weight $\lambda - \mu$ in $V(\lambda)$, and the value at $q = -1$ is the multiplicity of the weight $\lambda + \rho - \mu$ in the tensor product $V(\lambda) \otimes V(\rho)$. See Corollary 2.10. It is also related to Kazhdan-Lusztig polynomials when $\mathfrak{g}$ is of finite type (Corollary 3.30).

Our construction also has connections to deformations of Kostant's partition function. When $q = -1$ and λ is a strictly dominant weight, the Casselman-Shalika formula (0.5) gives a formula for multiplicity of the weight ν in the tensor product $V(\lambda-\rho)\otimes V(\rho)$ in terms of q-deformation of Kostant partition function, generalizing the result of [13, Theorem 1] to affine Kac-Moody algebras (See (3.24)). More precisely, we define $K_q^\infty(\mu)$, in a similar way as in [13], by

$$\sum_{\mu\in Q_+} K_q^\infty(\mu)\mathbf{z}^\mu = \prod_{\alpha\in\Delta^+} \left(\frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha}\right)^{\mathrm{mult}\,\alpha}.$$

Note that when $q = \infty$, $K_q^\infty(\mu)$ is the classical Kostant partition function. Then we have

$$\dim\left(V(\lambda-\rho)\otimes V(\rho)\right)_\nu = \sum_{w\in W} (-1)^{l(w)} K_{-1}^\infty(w\lambda - \nu).$$

Another application is deformation of arithmetical functions. Since the set of positive roots is infinite, the left-hand sides of (0.4) and (0.5) become infinite products. It leads to natural q-deformation of arithmetical functions such as multi-partition functions and Fourier coefficients of modular forms. We indicate one example here.

We define $\epsilon_{q,n}(k)$ as

$$\prod_{k=1}^\infty (1-q^{-1}t^k)^n = \sum_{k=0}^\infty \epsilon_{q,n}(k)t^k.$$

Note that $\epsilon_{1,n}(k)$ is a classical arithmetic function related to modular forms. For example, we have $\epsilon_{1,24}(k) = \tau(k+1)$, where $\tau(k)$ is the Ramanujan τ-function. Thus the function $\epsilon_{q,n}(k)$ should be considered as a q-deformation of the function $\epsilon_{1,n}(k)$.

For a multi-partition $\mathbf{p} = (\rho^{(1)}, \dots, \rho^{(n)}) \in \mathcal{P}(n)$, we define

$$p_{q,n}(k) = \sum_{\substack{\mathbf{p}\in\mathcal{P}(n)\\ |\mathbf{p}|=k}} (1-q^{-1})^{d(\mathbf{p})}, \qquad k \geq 1,$$

and set $p_{q,n}(0) = 1$. Here $|\mathbf{p}|$ is the weight of the multi-partition and the number $d(\mathbf{p})$ is defined in Section 1. Notice that if $q \to \infty$ and $k > 0$, the function $p_{\infty,n}(k)$ is nothing but the multi-partition function with n-components. In particular, $p_{\infty,1}(k) = p(k)$, the usual partition function. Hence we can think of $p_{q,n}(k)$ as a q-deformation of the multi-partition function.

It turned out that there are interesting relations among these q-deformations. We prove (Proposition 3.8)

$$\epsilon_{q,n}(k) = \sum_{r=0}^{k} \epsilon_{1,n}(r) p_{q,n}(k-r),$$

which yields an infinite family of q-polynomial identities. We also obtain "classical" identities by taking limits. When $n = 24$ and $q \to \infty$, the identity becomes a well-known recurrence formula for the Ramanujan τ-function:

$$0 = \sum_{r=0}^{k} \tau(r+1) p_{\infty,24}(k-r).$$

In fact, we prove another family of identities (Proposition 3.13) and obtain an intriguing characterization of the function $\epsilon_{q,n}(k)$. In Example 3.14, by taking $q = 1$, we write $\tau(k+1)$ as a sum of certain numbers arising from the structure of the affine Lie algebra of type $A_4^{(1)}$. To be precise, we have

$$\tau(k+1) = \lim_{q\to 1} \sum_{\mu\in Q_{+,\mathrm{cl}}} H_\rho(k\alpha_0 + \mu; q) \Big/ (1-q^{-1})^{10},$$

where $Q_{+,\mathrm{cl}}$ is the classical nonnegative root lattice of type A_4.

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1. Gindikin-Karpelevich Formula

Let $\mathfrak{g}$ be an untwisted affine Kac-Moody algebra over $\mathbb{C}$. We denote by $I = \{0, 1, \dots, n\}$ the set of indices for simple roots. Let W be the Weyl group. We keep almost all the notations in Sections 2 and 3 of [3]. However, we use v for the parameter of a quantum group and reserve q for another parameter. Whenever there is a discrepancy in notations, we will make it clear.

We fix $\mathbf{h} = (\dots, i_{-1}, i_0, i_1, \dots)$ as in Section 3.1 in [3]. Then for any integers $m < k$, the product $s_{i_m} s_{i_{m+1}} \cdots s_{i_k} \in W$ is a reduced expression, so is the product $s_{i_k} s_{i_{k-1}} \cdots s_{i_m} \in W$. We set

$$\beta_k = \begin{cases} s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \le 0, \\ s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k > 0, \end{cases}$$

and define

$$\mathscr{R}(k) = \{\beta_0, \beta_{-1}, \dots, \beta_k\} \text{ for } k \le 0 \quad \text{and} \quad \mathscr{R}(k) = \{\beta_1, \beta_2, \dots, \beta_k\} \text{ for } k > 0.$$

Let $T_i = T''_{i,1}$ be the automorphism of $\mathbf{U}$ as in Section 37.1.3. of [20], and let

$$\mathbf{c}_+ = (c_0, c_{-1}, c_{-2}, \dots) \in \mathbb{N}^{\mathbb{Z}_{\le 0}} \quad \text{and} \quad \mathbf{c}_- = (c_1, c_2, \dots) \in \mathbb{N}^{\mathbb{Z}_{>0}}$$

be functions (or sequences) that are almost everywhere zero. We denote by $\mathscr{C}_>$ (resp. by $\mathscr{C}_<$) the set of such functions $\mathbf{c}_+$ (resp. $\mathbf{c}_-$). Then we define

$$E_{\mathbf{c}_+} = E_{i_0}^{(c_0)} T_{i_0}^{-1}\left(E_{i_{-1}}^{(c_{-1})}\right) T_{i_0}^{-1} T_{i_{-1}}^{-1}\left(E_{i_{-2}}^{(c_{-2})}\right) \cdots$$

and

$$E_{\mathbf{c}_-} = \cdots T_{i_1} T_{i_2}\left(E_{i_3}^{(c_3)}\right) T_{i_1}\left(E_{i_2}^{(c_2)}\right) E_{i_1}^{(c_1)}.$$

We set

$$B(k) = \begin{cases} \{E_{\mathbf{c}_+} : c_m = 0 \text{ for } m < k\} & \text{for } k \le 0, \\ \{E_{\mathbf{c}_-} : c_m = 0 \text{ for } m > k\} & \text{for } k > 0. \end{cases}$$

We denote by $\mathbf{B}$ the Kashiwara-Lusztig's canonical bases for $\mathbf{U}^+$, the positive part of the quantum affine algebra.

Proposition 1.1. [2, 3] *For each $E_{\mathbf{c}_+} \in B(k)$, $k \le 0$ (resp. $E_{\mathbf{c}_-} \in B(k)$, $k > 0$), there exists a unique $b \in \mathbf{B}$ such that*

$$b \equiv E_{\mathbf{c}_+} \text{ (resp. } E_{\mathbf{c}_-}) \mod v^{-1}\mathbb{Z}[v^{-1}]. \tag{1.2}$$

We denote by $\mathbf{B}(k)$ the subset of $\mathbf{B}$ corresponding to $B(k)$ as in the above theorem. Then we define the map $\phi : \mathbf{B}(k) \to \mathscr{C}_>$ for $k \le 0$ (resp. $\mathscr{C}_<$ for $k > 0$) to be $b \mapsto \mathbf{c}_+$ (resp. $\mathbf{c}_-$) such that the condition (1.2) holds. For an element $\mathbf{c}_+ = (c_0, c_{-1}, \dots) \in \mathscr{C}_>$ (resp. $\mathbf{c}_- = (c_1, c_2, \dots) \in \mathscr{C}_>$), we define $d(\mathbf{c}_+)$ (resp. $d(\mathbf{c}_-)$) to be the number of nonzero c_i's.

Proposition 1.3. *For each $k \in \mathbb{Z}$, we have*

$$\prod_{\alpha \in \mathscr{R}(k)} \frac{1 - q^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha} = \sum_{b \in \mathbf{B}(k)} (1 - q^{-1})^{d(\phi(b))} \mathbf{z}^{\mathrm{wt}(b)}. \tag{1.4}$$

Proof. First we assume $k > 0$ and use induction on k. If $k = 1$, then the identity (1.4) is easily verified. Now, using an induction argument, we obtain

$$\begin{aligned}
\prod_{\alpha \in \mathscr{R}(k)} \frac{1 - q^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha} &= \left(\prod_{\alpha \in \mathscr{R}(k-1)} \frac{1 - q^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha} \right) \frac{1 - q^{-1}\mathbf{z}^{\beta_k}}{1 - \mathbf{z}^{\beta_k}} \\
&= \left(\sum_{b \in \mathbf{B}(k-1)} (1 - q^{-1})^{d(\phi(b))} \mathbf{z}^{\mathrm{wt}(b)} \right) \left(1 + \sum_{j \ge 1} (1 - q^{-1}) \mathbf{z}^{j\beta_k} \right) \\
&= \sum_{b \in \mathbf{B}(k-1)} (1 - q^{-1})^{d(\phi(b))} \mathbf{z}^{\mathrm{wt}(b)} + \sum_{j \ge 1} \sum_{b \in \mathbf{B}(k-1)} (1 - q^{-1})^{d(\phi(b))+1} \mathbf{z}^{\mathrm{wt}(b) + j\beta_k}.
\end{aligned}$$

On the other hand, since $b' \in \mathbf{B}(k)$ satisfies

$$b' \equiv b\, T_{i_1} T_{i_2} \cdots T_{i_k} \left(E_k^{(j)} \right) \mod v^{-1}\mathbb{Z}[v^{-1}]$$

for unique $b \in \mathbf{B}(k-1)$ and $j \ge 0$, we can write $\mathbf{B}(k)$ as a disjoint union

$$\mathbf{B}(k) = \bigcup_{j \ge 0} \{ b' \in \mathbf{B}(k) \mid \phi(b') = (c_1, \dots, c_{k-1}, j, 0, 0, \dots),\ c_i \in \mathbb{N} \}.$$

Now it is clear that

$$\begin{aligned}
&\sum_{b \in \mathbf{B}(k)} (1 - q^{-1})^{d(\phi(b))} \mathbf{z}^{\mathrm{wt}(b)} \\
= &\sum_{b \in \mathbf{B}(k-1)} (1 - q^{-1})^{d(\phi(b))} \mathbf{z}^{\mathrm{wt}(b)} + \sum_{j \ge 1} \sum_{b \in \mathbf{B}(k-1)} (1 - q^{-1})^{d(\phi(b))+1} \mathbf{z}^{\mathrm{wt}(b) + j\beta_k}.
\end{aligned}$$

This completes the proof of the case $k > 0$.

The case $k \le 0$ can be proved in a similar way through a downward induction. □

We set

$$\mathscr{R}_> = \bigcup_{k \le 0} \mathscr{R}(k) \quad \text{and} \quad \mathscr{R}_< = \bigcup_{k > 0} \mathscr{R}(k).$$

Similarly, we put

$$\mathbf{B}_{>} = \bigcup_{k\le 0} \mathbf{B}(k) \quad \text{and} \quad \mathbf{B}_{<} = \bigcup_{k>0} \mathbf{B}(k).$$

Corollary 1.5. *We have*

$$\prod_{\alpha\in\mathscr{R}_{>}} \frac{1-q^{-1}\mathbf{z}^{\alpha}}{1-\mathbf{z}^{\alpha}} = \sum_{b\in\mathbf{B}_{>}} (1-q^{-1})^{d(\phi(b))}\mathbf{z}^{\mathrm{wt}(b)}. \tag{1.6}$$

The same identity is true if $\mathscr{R}_{>}$ and $\mathbf{B}_{>}$ are replaced with $\mathscr{R}_{<}$ and $\mathbf{B}_{<}$, respectively.

Let $\mathbf{c}_0 = (\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)})$ be a multi-partition with n components, *i.e.* each component $\rho^{(i)}$ is a partition. We denote by $\mathcal{P}(n)$ the set of all multi-partitions with n components. Let $S_{\mathbf{c}_0}$ be defined as in [3] (p. 352) and set

$$B_0 = \{S_{\mathbf{c}_0} \mid \mathbf{c}_0 \in \mathcal{P}(n)\}.$$

Proposition 1.7. [2, 3] *For each $S_{\mathbf{c}_0} \in B_0$, there exists a unique $b \in \mathbf{B}$ such that*

$$b \equiv S_{\mathbf{c}_0} \mod v^{-1}\mathbb{Z}[v^{-1}]. \tag{1.8}$$

We denote by $\mathbf{B}_0$ the subset of $\mathbf{B}$ corresponding to B_0. Using the same notation ϕ as we used for $\mathbf{B}(k)$, we define a function $\phi : \mathbf{B}_0 \to \mathcal{P}(n)$, $b \mapsto \mathbf{c}_0$, such that the condition (1.8) is satisfied.

For a partition $\mathbf{p} = (1^{m_1}2^{m_2}\cdots r^{m_r}\cdots)$, we define

$$d(\mathbf{p}) = \#\{r \mid m_r \neq 0\} \quad \text{and} \quad |\mathbf{p}| = m_1 + 2m_2 + 3m_3 + \cdots .$$

Then for a multi-partition $\mathbf{c}_0 = (\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)}) \in \mathcal{P}(n)$, we set

$$d(\mathbf{c}_0) = d(\rho^{(1)}) + d(\rho^{(2)}) + \cdot + d(\rho^{(n)}).$$

We obtain from the definition of $S_{\mathbf{c}_0}$ that if $\phi(b) = \mathbf{c}_0$ then

$$\mathrm{wt}(b) = |\mathbf{c}_0|\delta,$$

where $|\mathbf{c}_0| = |\rho^{(1)}| + \cdots + |\rho^{(n)}|$ is the weight of the multi-partition $\mathbf{c}_0$.

Proposition 1.9. *We have*

$$\prod_{\alpha\in\Delta^{+}_{\mathrm{im}}} \left(\frac{1-q^{-1}\mathbf{z}^{\alpha}}{1-\mathbf{z}^{\alpha}}\right)^{\mathrm{mult}\,\alpha} = \prod_{k=1}^{\infty} \left(\frac{1-q^{-1}\mathbf{z}^{k\delta}}{1-\mathbf{z}^{k\delta}}\right)^{n} = \sum_{b\in\mathbf{B}_0} (1-q^{-1})^{d(\phi(b))}\mathbf{z}^{\mathrm{wt}(b)}, \tag{1.10}$$

where Δ^{+}_{im} is the set of positive imaginary roots of $\mathfrak{g}$.

Proof. The first equality follows from the facts $\Delta^+_{\mathrm{im}} = \{\delta, 2\delta, 3\delta, \dots\}$ and $\mathrm{mult}(k\delta) = n$ for all $k = 1, 2, \dots$. Now we consider the second equality and assume $n = 1$. Then we have

$$\prod_{k=1}^{\infty} \left(\frac{1 - q^{-1}\mathbf{z}^{k\delta}}{1 - \mathbf{z}^{k\delta}} \right) = \prod_{k=1}^{\infty} \left(1 + \sum_{j=1}^{\infty} (1 - q^{-1}) \mathbf{z}^{jk\delta} \right). \tag{1.11}$$

We consider the generating function of the partition function $p(m)$:

$$\sum_{m=0}^{\infty} p(m) \mathbf{z}^{m\delta} = \prod_{k=1}^{\infty} \left(1 + \sum_{j=1}^{\infty} \mathbf{z}^{jk\delta} \right) = \sum_{\rho^{(1)} \in \mathcal{P}(1)} \mathbf{z}^{|\rho^{(1)}|\delta} = \sum_{b \in \mathbf{B}_0} \mathbf{z}^{\mathrm{wt}(b)}. \tag{1.12}$$

Comparing (1.11) and (1.12), we see that if we expand the product in the right-hand side of (1.11) into a sum, the coefficient of $\mathbf{z}^{|\rho^{(1)}|\delta}$ will be a power of $(1 - q^{-1})$ and that the exponent of $(1 - q^{-1})$ is exactly the number $d(\rho^{(1)})$. Therefore, we obtain

$$\begin{aligned} \prod_{k=1}^{\infty} \left(\frac{1 - q^{-1}\mathbf{z}^{k\delta}}{1 - \mathbf{z}^{k\delta}} \right) &= \sum_{\rho^{(1)} \in \mathcal{P}(1)} (1 - q^{-1})^{d(\rho^{(1)})} \mathbf{z}^{|\rho^{(1)}|\delta} \\ &= \sum_{b \in \mathbf{B}_0} (1 - q^{-1})^{d(b)} \mathbf{z}^{\mathrm{wt}(b)}. \end{aligned}$$

Next we assume that $n = 2$. Then we have

$$\begin{aligned} &\prod_{k=1}^{\infty} \left(\frac{1 - q^{-1}\mathbf{z}^{k\delta}}{1 - \mathbf{z}^{k\delta}} \right)^2 \\ = &\left(\sum_{\rho^{(1)} \in \mathcal{P}(1)} (1 - q^{-1})^{d(\rho^{(1)})} \mathbf{z}^{|\rho^{(1)}|\delta} \right) \left(\sum_{\rho^{(2)} \in \mathcal{P}(1)} (1 - q^{-1})^{d(\rho^{(2)})} \mathbf{z}^{|\rho^{(2)}|\delta} \right) \\ = &\sum_{(\rho^{(1)}, \rho^{(2)}) \in \mathcal{P}(2)} (1 - q^{-1})^{d(\rho^{(1)}) + d(\rho^{(2)})} \mathbf{z}^{\left(|\rho^{(1)}| + |\rho^{(2)}|\right)\delta} \\ = &\sum_{b \in \mathbf{B}_0} (1 - q^{-1})^{d(b)} \mathbf{z}^{\mathrm{wt}(b)}. \end{aligned}$$

It is now clear that this argument naturally generalizes to the case $n > 2$. □

Let us consider the correction factor A in (0.3). We will make a modification of the formula (1.10) to write A as a sum over $\mathbf{B}_0$ in the case when the underlying classical Lie algebra $\mathfrak{g}_{\mathrm{cl}}$ is

simply-laced. For a partition $\mathbf{p} = (1^{m_1}2^{m_2}\cdots)$ and $d_i \in \mathbb{N}$, we define

$$Q_{d_i}(\mathbf{p}, j) = \begin{cases} (1-q)q^{-(d_i+1)m_j} & \text{if } m_j \neq 0, \\ 1 & \text{if } m_j = 0, \end{cases} \quad \text{and} \quad Q_{d_i}(\mathbf{p}) = \prod_{j=1}^{\infty} Q_{d_i}(\mathbf{p}, j).$$

For a multi-partition $\mathbf{p} = (\rho^{(1)}, \ldots, \rho^{(n)})$ and $d_i \in \mathbb{N}$, $i = 1, \ldots, n$, we define

$$Q_{d_1,\ldots,d_n}(\mathbf{p}) = \prod_{i=1}^{n} \quad Q_{d_i}(\rho^{(i)}).$$

Then we obtain the following.

Corollary 1.13. *Assume that $\mathfrak{g}_{\mathrm{cl}}$ is simply-laced. Then we have*

$$A = \prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{1 - q^{-d_i}\mathbf{z}^{j\delta}}{1 - q^{-d_i-1}\mathbf{z}^{j\delta}} = \sum_{b \in \mathbf{B}_0} Q(\phi(b))\mathbf{z}^{\mathrm{wt}(b)},$$

where d_i's are the exponents of $\mathfrak{g}_{\mathrm{cl}}$ and we write $Q(\mathbf{p}) = Q_{d_1,\ldots,d_n}(\mathbf{p})$.

Proof. The first equality is a result in [4]. The second equality can be obtained using a similar argument as in the proof of Proposition 1.9. □

Let $\mathscr{C} = \mathscr{C}_> \times \mathcal{P}(n) \times \mathscr{C}_<$ as in [3].

Theorem 1.14. [2, 3] *There is a bijection between the sets $\mathbf{B}$ and $\mathscr{C}$ such that for each $\mathbf{c} = (\mathbf{c}_+, \mathbf{c}_0, \mathbf{c}_-) \in \mathscr{C}$, there exists a unique $b \in \mathbf{B}$ such that*

$$b \equiv E_{\mathbf{c}_+} S_{\mathbf{c}_0} E_{\mathbf{c}_-} \mod v^{-1}\mathbb{Z}[v^{-1}]. \tag{1.15}$$

Then we naturally extend the function ϕ to a bijection of $\mathbf{B}$ onto $\mathscr{C}$ and the number $d(\mathbf{c})$ is also defined by $d(\mathbf{c}) = d(\mathbf{c}_+) + d(\mathbf{c}_0) + d(\mathbf{c}_-)$ for each $\mathbf{c} \in \mathscr{C}$.

Theorem 1.16. *We have*

$$\prod_{\alpha \in \Delta^+} \left(\frac{1 - q^{-1}\mathbf{z}^{\alpha}}{1 - \mathbf{z}^{\alpha}} \right)^{\mathrm{mult}\,\alpha} = \sum_{b \in \mathbf{B}} (1 - q^{-1})^{d(\phi(b))}\mathbf{z}^{\mathrm{wt}(b)}. \tag{1.17}$$

Proof. Recall that $\Delta^+ = \Delta^+_{\mathrm{re}} \cup \Delta^+_{\mathrm{im}}$, $\Delta^+_{\mathrm{re}} = \mathscr{R}_> \cup \mathscr{R}_<$ and mult $\alpha = 1$ for $\alpha \in \Delta^+_{\mathrm{re}}$. Then the identity of the theorem follows from Corollary 1.5, Proposition 1.9 and Theorem 1.14. □

2. Casselman-Shalika Formula

For the functions $\mathbf{c}_+ = (c_0, c_{-1}, c_{-2}, \dots) \in \mathscr{C}_>$ and $\mathbf{c}_- = (c_1, c_2, \dots) \in \mathscr{C}_<$, we define

$$|\mathbf{c}_+| = c_0 + c_{-1} + c_{-2} + \cdots \quad \text{and} \quad |\mathbf{c}_-| = c_1 + c_2 + \cdots .$$

For a multi-partition $\mathbf{c}_0 = (\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)}) \in \mathcal{P}(n)$, we set $|\mathbf{c}_0| = |\rho^{(1)}| + \cdots + |\rho^{(n)}|$, as we did in the previous section.

Using similar arguments in the previous section, we obtain the following identities.

Proposition 2.1.

(1) *We have for each $k \in \mathbb{Z}$,*

$$\prod_{\alpha \in \mathscr{R}(k)} (1 - q^{-1}\mathbf{z}^{\alpha})^{-1} = \sum_{b \in \mathbf{B}(k)} q^{-|\phi(b)|} \mathbf{z}^{\mathrm{wt}(b)}.$$

(2)

$$\prod_{\alpha \in \mathscr{R}_>} (1 - q^{-1}\mathbf{z}^{\alpha})^{-1} = \sum_{b \in \mathbf{B}_>} q^{-|\phi(b)|} \mathbf{z}^{\mathrm{wt}(b)}.$$

The same identity is true if $\mathscr{R}_>$ and $\mathbf{B}_>$ are replaced with $\mathscr{R}_<$ and $\mathbf{B}_<$, respectively.

(3)

$$\prod_{\alpha \in \Delta^+_{\mathrm{im}}} \left(1 - q^{-1}\mathbf{z}^{\alpha}\right)^{-\mathrm{mult}\,\alpha} = \prod_{k=1}^{\infty} \left(1 - q^{-1}\mathbf{z}^{k\delta}\right)^{-n} = \sum_{b \in \mathbf{B}_0} q^{-|\phi(b)|} \mathbf{z}^{\mathrm{wt}(b)}.$$

(4)

$$\prod_{\alpha \in \Delta^+} \left(1 - q^{-1}\mathbf{z}^{\alpha}\right)^{-\mathrm{mult}\,\alpha} = \sum_{b \in \mathbf{B}} q^{-|\phi(b)|} \mathbf{z}^{\mathrm{wt}(b)}.$$

Let $P_+ = \{\lambda \in P | \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$. Recall that the irreducible $\mathfrak{g}$-module $V(\lambda)$ is integrable if and only if $\lambda \in P_+$ ([14], Lemma 10.1).

Definition 2.2. Let $\lambda \in P_+$. We define $H_\lambda(\cdot; q) : Q_+ \to \mathbb{Z}[q^{-1}]$ using the generating series

$$\begin{aligned}\sum_{\mu \in Q_+} H_\lambda(\mu; q) \mathbf{z}^{\lambda - \mu} &= \sum_{w \in W} (-1)^{\ell(w)} \sum_{b \in \mathbf{B}} (1 - q^{-1})^{d(\phi(b))} \mathbf{z}^{w\lambda - \mathrm{wt}(b)} \\ &= \left(\sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w\lambda} \right) \left(\sum_{b \in \mathbf{B}} (1 - q^{-1})^{d(\phi(b))} \mathbf{z}^{-\mathrm{wt}(b)} \right),\end{aligned}$$

and we write

$$\chi_q(V(\lambda)) = \sum_{\mu \in Q_+} H_\lambda(\mu; q)\ \mathbf{z}^{\lambda - \mu}.$$

We denote by $\chi(V(\lambda))$ the usual character of $V(\lambda)$. We have the element $d \in \mathfrak{h}$ such that $\alpha_0(d) = 1$ and $\alpha_j(d) = 0$, $j \in I \setminus \{0\}$. We define $\rho \in \mathfrak{h}^*$ as in [14, chapter 6] by $\rho(h_j) = 1$, $j \in I$ and $\rho(d) = 0$. By the Weyl-Kac character formula,

$$\frac{\displaystyle\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w(\lambda+\rho)-\rho}}{\displaystyle\prod_{\alpha\in\Delta^+}(1-\mathbf{z}^{-\alpha})^{\operatorname{mult}\alpha}} = \chi(V(\lambda)).$$

In particular, if $\lambda = 0$, then

$$\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w\rho} = \mathbf{z}^{\rho}\prod_{\alpha\in\Delta^+}(1-\mathbf{z}^{-\alpha})^{\operatorname{mult}\alpha}.$$

By Theorem 1.16,

$$\sum_{b\in\mathbf{B}}(1-q^{-1})^{d(\phi(b))}\mathbf{z}^{-\mathrm{wt}(b)} = \prod_{\alpha\in\Delta^+}\left(\frac{1-q^{-1}\mathbf{z}^{-\alpha}}{1-\mathbf{z}^{-\alpha}}\right)^{\operatorname{mult}\alpha}.$$

Thus we obtain

$$\begin{aligned}\chi_q(V(\rho)) &= \left(\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w\rho}\right)\left(\sum_{b\in\mathbf{B}}(1-q^{-1})^{d(\phi(b))}\mathbf{z}^{-\mathrm{wt}(b)}\right)\\ &= \mathbf{z}^{\rho}\prod_{\alpha\in\Delta^+}(1-\mathbf{z}^{-\alpha})^{\operatorname{mult}\alpha}\prod_{\alpha\in\Delta^+}\left(\frac{1-q^{-1}\mathbf{z}^{-\alpha}}{1-\mathbf{z}^{-\alpha}}\right)^{\operatorname{mult}\alpha}\\ &= \mathbf{z}^{\rho}\prod_{\alpha\in\Delta^+}(1-q^{-1}\mathbf{z}^{-\alpha})^{\operatorname{mult}\alpha}.\end{aligned}$$

Therefore we have proved the following.

$$\chi_q(V(\rho)) = \mathbf{z}^{\rho}\prod_{\alpha\in\Delta^+}(1-q^{-1}\mathbf{z}^{-\alpha})^{\operatorname{mult}\alpha}. \tag{2.3}$$

When $q = -1$ in (2.3), we have the following identity by [14, Exercise 10.1].

Lemma 2.4.

$$\chi_{-1}(V(\rho)) = \mathbf{z}^{\rho}\prod_{\alpha\in\Delta^+}(1+\mathbf{z}^{-\alpha})^{\operatorname{mult}\alpha} = \chi(V(\rho)).$$

Remark 2.5. By Definition 2.2,

$$\chi_{-1}(V(\rho)) = \sum_{\mu\in Q_+}H_\rho(\mu;-1)\mathbf{z}^{\rho-\mu} = \mathbf{z}^{\rho}\prod_{\alpha\in\Delta^+}(1+\mathbf{z}^{-\alpha})^{\operatorname{mult}\alpha}.$$

Therefore, if $H_\rho(\mu; -1) \neq 0$, $\rho - \mu$ must be a weight of $V(\rho)$ and $H_\rho(\mu; -1)$ is the multiplicity of $\rho - \mu$ in $V(\rho)$.

Now we have the following proposition which is an affine analogue of the Casselman-Shalika formula.

Proposition 2.6.

$$\chi_q(V(\lambda+\rho)) = \chi(V(\lambda))\chi_q(V(\rho)). \tag{2.7}$$

Proof. By Definition 2.2 and Theorem 1.16,

$$\chi_q(V(\lambda+\rho)) = \left(\sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w(\lambda+\rho)}\right) \prod_{\alpha \in \Delta^+} \left(\frac{1-q^{-1}\mathbf{z}^{-\alpha}}{1-\mathbf{z}^{-\alpha}}\right)^{\mathrm{mult}\,\alpha}.$$

By the Weyl-Kac character formula and (2.3), the right hand side is $\chi(V(\lambda))\chi_q(V(\rho))$. $\square$

Remark 2.8. When $q = 1$, we see that $\chi_1(V(\lambda+\rho))\mathbf{z}^{-\rho}$ is the numerator of the Weyl-Kac character formula. Hence we can think of (2.7) as a q-deformation of Weyl-Kac character formula.

At special values of q, the formal sum $\chi_q(V(\lambda+\rho))$ becomes characters of some representations as you can see in the following corollary.

Corollary 2.9.

(1) *When $q = \infty$, we have*

$$\chi_\infty(V(\lambda+\rho)) = \mathbf{z}^\rho \chi(V(\lambda)).$$

Hence we may consider $\chi_q(V(\lambda+\rho))\mathbf{z}^{-\rho}$ as a q-deformation of $\chi(V(\lambda))$.

(2) *When $q = -1$, we obtain*

$$\chi_{-1}(V(\lambda+\rho)) = \chi(V(\lambda))\chi(V(\rho)) = \chi(V(\lambda) \otimes V(\rho)).$$

Proof. (1) The identity is true since $\chi_\infty(V(\rho)) = \mathbf{z}^\rho$. (2) By putting $q = -1$ in (2.7), the identity follows from Lemma 2.4. $\square$

We also obtain representation-theoretic meaning of special values of $H_{\lambda+\rho}(\mu; q)$.

Corollary 2.10.

(1) *The value $H_{\lambda+\rho}(\mu; \infty)$ is the multiplicity of the weight $\lambda - \mu$ in $V(\lambda)$.*

(2) *The value* $H_{\lambda+\rho}(\mu;-1)$ *is the multiplicity of the weight* $\lambda+\rho-\mu$ *in the tensor product* $V(\lambda)\otimes V(\rho)$.

Proof. (1) By Definition 2.2 and from Corollary 2.9 (1), we have

$$\sum_{\mu\in Q_+} H_{\lambda+\rho}(\mu;\infty)\mathbf{z}^{\lambda-\mu} = \mathbf{z}^{-\rho}\chi_\infty(V(\lambda+\rho)) = \chi(V(\lambda)).$$

This proves the part (1).

(2) We obtain from Corollary 2.9 (2)

$$\chi_{-1}(V(\lambda+\rho)) = \sum_{\mu\in Q_+} H_{\lambda+\rho}(\mu;-1)\mathbf{z}^{\lambda+\rho-\mu} = \chi(V(\lambda)\otimes V(\rho)),$$

which proves the part (2). □

Before we further investigate the implication of the Casselman-Shalika formula (2.7), we need the following lemma.

Lemma 2.11. *Assume that* $\lambda_1,\lambda_2\in P_+$. *Then the set of weights of* $V(\lambda_1)\otimes V(\lambda_2)$ *is the same as that of* $V(\lambda_1+\lambda_2)$.

Proof. Suppose that $\lambda_1,\lambda_2\in P_+$. Let $V(\lambda_1)$ and $V(\lambda_2)$ be the integrable highest weight modules with highest weights λ_1 and λ_2, respectively. By [14, p. 211], $V(\lambda_1+\lambda_2)$ occurs in $V(\lambda_1)\otimes V(\lambda_2)$ with multiplicity one. Hence it is enough to prove that any weight of $V(\lambda_1)\otimes V(\lambda_2)$ is a weight of $V(\lambda_1+\lambda_2)$.

If V_1 and V_2 are modules in the category $\mathcal{O}$, then the weight space of $(V_1\otimes V_2)_\mu$ for $\mu\in\mathfrak{h}^*$, is given by

$$(V_1\otimes V_2)_\mu = \sum_{\nu\in\mathfrak{h}^*}(V_1)_\nu\otimes(V_2)_{\mu-\nu}.$$

Hence weights of $V(\lambda_1)\otimes V(\lambda_2)$ are of the form $\mu_1+\mu_2$, where μ_1 and μ_2 are weights of $V(\lambda_1)$ and $V(\lambda_2)$, respectively. Furthermore, since $V(\lambda_1)\otimes V(\lambda_2)$ is completely reducible, a weight $\mu_1+\mu_2$ of $V(\lambda_1)\otimes V(\lambda_2)$ is a weight of the module $V(\lambda)$ for some $\lambda\in P_+$, that appears in the decomposition of $V(\lambda_1)\otimes V(\lambda_2)$.

It follows from Corollary 10.1 in [14] that we can choose $w\in W$ such that $w(\mu_1+\mu_2)\in P_+$. Then, by Proposition 11.2 in [14], we need only to show that $w(\mu_1+\mu_2)$ is nondegenerate with respect to $\lambda_1+\lambda_2$. By Lemma 11.2 in [14], $w\mu_1$ and $w\mu_2$ are nondegenerate with respect to λ_1 and λ_2, respectively. Now, from the definition of nondegeneracy [14, p.190], we see that $w\mu_1+w\mu_2$ is nondegenerate with respect to $\lambda_1+\lambda_2$. □

Now we use crystal bases, namely, bases at $v = 0$, since they behave nicely under tensor products. Let $\mathfrak{B}_\lambda$ be the crystal basis associated to a dominant integral weight $\lambda \in P_+$. We choose $G_\rho(\cdot; q) : \mathfrak{B}_\rho \to \mathbb{Z}[q^{-1}]$ by assigning any element of $\mathbb{Z}[q^{-1}]$ to each $b \in \mathfrak{B}_\rho$ so that

$$H_\rho(\mu; q) = \sum_{\substack{b \in \mathfrak{B}_\rho \\ \mathrm{wt}(b) = \rho - \mu}} G_\rho(b; q). \tag{2.12}$$

By Remark 2.5, it is enough to consider $\mu \in Q_+$ such that $\rho - \mu$ is a weight of $b \in \mathfrak{B}_\rho$.

Using the function $G_\rho(\cdot; q)$, we can rewrite Casselman-Shalika formula in Proposition 2.6 in a familiar form:

Corollary 2.13.
(2.14)

$$\sum_{\mu \in Q_+} H_{\lambda+\rho}(\mu; q)\, \mathbf{z}^{\lambda+\rho-\mu} = \chi(V(\lambda))\mathbf{z}^\rho \prod_{\alpha \in \Delta^+} (1 - q^{-1}\mathbf{z}^{-\alpha})^{\mathrm{mult}\,\alpha} = \sum_{b' \otimes b \in \mathfrak{B}_\lambda \otimes \mathfrak{B}_\rho} G_\rho(b; q)\, \mathbf{z}^{\mathrm{wt}(b' \otimes b)}.$$

Proof. The first equality is obvious from (2.3) and Proposition 2.6. For the second equality, we obtain

$$\begin{aligned} & \chi(V(\lambda))\mathbf{z}^\rho \prod_{\alpha \in \Delta^+} (1 - q^{-1}\mathbf{z}^{-\alpha})^{\mathrm{mult}\,\alpha} = \chi(V(\lambda))\chi_q(V(\rho)) \\ = & \left(\sum_{b' \in \mathfrak{B}_\lambda} \mathbf{z}^{\mathrm{wt}(b')} \right) \left(\sum_{\mu \in Q_+} H_\rho(\mu; q)\, \mathbf{z}^{\rho-\mu} \right) = \left(\sum_{b' \in \mathfrak{B}_\lambda} \mathbf{z}^{\mathrm{wt}(b')} \right) \left(\sum_{b \in \mathfrak{B}_\rho} G_\rho(b; q)\mathbf{z}^{\mathrm{wt}(b)} \right) \\ = & \sum_{b' \otimes b \in \mathfrak{B}_\lambda \otimes \mathfrak{B}_\rho} G_\rho(b; q)\, \mathbf{z}^{\mathrm{wt}(b' \otimes b)}. \end{aligned}$$

□

The following proposition provides useful information on $H_{\lambda+\rho}(\mu; q) \in \mathbb{Z}[q^{-1}]$.

Proposition 2.15. *Assume that $\lambda \in P_+$. Then we have $H_{\lambda+\rho}(\mu; q)$ is a nonzero polynomial if and only if $\lambda + \rho - \mu$ is a weight of $V(\lambda + \rho)$.*

Proof. We obtain from (2.14) that if $H_{\lambda+\rho}(\mu; q) \neq 0$ then $\lambda + \rho - \mu$ is a weight of $V(\lambda) \otimes V(\rho)$. Then $\lambda + \rho - \mu$ is a weight of $V(\lambda + \rho)$ by Lemma 2.11. Conversely, assume that $\lambda + \rho - \mu$ is a weight of $V(\lambda + \rho)$, so a weight of $V(\lambda) \otimes V(\rho)$. By Corollary 2.9 (2),

$$\sum_{\mu' \in Q_+} H_{\lambda+\rho}(\mu'; -1)\mathbf{z}^{\lambda+\rho-\mu'} = \chi(V(\lambda) \otimes V(\rho)).$$

Since $\lambda+\rho-\mu$ is a weight of $V(\lambda)\otimes V(\rho)$, the coefficient $H_{\lambda+\rho}(\mu;-1)\neq 0$. Then $H_{\lambda+\rho}(\mu;q)$ is a nonzero polynomial. $\square$

3. Applications

We give several applications of our formulas to q-deformation of (multi-)partition functions and modular forms, and Kostant's function and multiplicity formula. We also obtain formulas for $H_\lambda(\mu;q)$.

3.1. **multi-partition functions and modular forms.** We will write $\mathcal{P}=\mathcal{P}(1)$. For a partition $\mathbf{p}=(1^{m_1}2^{m_2}\cdots r^{m_r}\cdots)\in\mathcal{P}$, we define

$$\kappa_q(\mathbf{p})=\begin{cases}(-q^{-1})^{\sum m_r} & \text{if } m_r=0 \text{ or } 1 \text{ for all } r,\\ 0 & \text{otherwise.}\end{cases}$$

We define for $k\geq 1$

$$\epsilon_q(k)=\sum_{\substack{\mathbf{p}\in\mathcal{P}\\ |\mathbf{p}|=k}}\kappa_q(\mathbf{p}),$$

and set $\epsilon_q(0)=1$. For example, we have $\epsilon_q(5)=2q^{-2}-q^{-1}$ and $\epsilon_q(6)=-q^{-3}+2q^{-2}-q^{-1}$.

From the definitions, we have

$$\prod_{k=1}^{\infty}(1-q^{-1}t^k)=1+\sum_{\mathbf{p}\in\mathcal{P}}\kappa_q(\mathbf{p})t^{|\mathbf{p}|}=1+\sum_{k=1}^{\infty}\epsilon_q(k)t^k.$$

Then it follows from Euler's Pentagonal Number Theorem that when $q=1$, we have

$$\epsilon_1(k)=\begin{cases}(-1)^m & \text{if } k=\frac{1}{2}m(3m\pm 1),\\ 0 & \text{otherwise.}\end{cases}\tag{3.1}$$

We also define for $k\geq 1$

$$p_q(k)=\sum_{\substack{\mathbf{p}\in\mathcal{P}\\ |\mathbf{p}|=k}}(1-q^{-1})^{d(\mathbf{p})},$$

where $d(\mathbf{p})$ is the same as in the previous sections, and we set $p_q(0)=1$. Note that if $k>0$, $p_\infty(k)=p(k)$. Hence we can think of $p_q(k)$ as a q-deformation of the partition function.

Proposition 3.2. *If $k > 0$, then*

$$\epsilon_q(k) - p_q(k) = \sum_{m=1}^{\infty} (-1)^m \left\{ p_q(k - \frac{1}{2}m(3m-1)) + p_q(k - \frac{1}{2}m(3m+1)) \right\}, \tag{3.3}$$

where we define $p_q(M) = 0$ for all negative integer M.

Proof. We put $n = 1$ in Proposition 1.9 and obtain

$$\prod_{k=1}^{\infty}(1 - q^{-1}\mathbf{z}^{k\delta}) = \left(\sum_{\mathbf{p}\in\mathcal{P}} (1-q^{-1})^{d(\mathbf{p})} \mathbf{z}^{|\mathbf{p}|\delta} \right) \prod_{k=1}^{\infty}(1 - \mathbf{z}^{k\delta}).$$

After the change of variables $\mathbf{z}^{\delta} = t$, we obtain

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} \epsilon_q(k) t^k &= \prod_{k=1}^{\infty}(1 - q^{-1}t^k) \\ &= \left(\sum_{\mathbf{p}\in\mathcal{P}} (1-q^{-1})^{d(\mathbf{p})} t^{|\mathbf{p}|} \right) \prod_{k=1}^{\infty}(1 - t^k) \\ &= \left(1 + \sum_{k=1}^{\infty} p_q(k) t^k \right) \left(1 + \sum_{m=1}^{\infty} (-1)^m \left\{ t^{\frac{1}{2}m(3m-1)} + t^{\frac{1}{2}m(3m+1)} \right\} \right), \end{aligned}$$

where we use the definition of $p_q(k)$ and (3.1) in the last equality. We obtain the identity (3.3) by expanding the product and equating the coefficient of t^k with $\epsilon_q(k)$. □

As a corollary of the proof of Proposition 3.2, we obtain the following.

Corollary 3.4. *Let $(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$. Then*

$$\sum_{n=0}^{\infty} \frac{(q^{-1};t)_n}{(t;t)_n} \, t^n = \sum_{k=0}^{\infty} p_q(k) t^k.$$

Proof. By the q-binomial theorem,

$$\prod_{k=1}^{\infty}(1 - q^{-1}t^k) = \left(\sum_{n=0}^{\infty} \frac{(q^{-1};t)_n}{(t;t)_n} \, t^n \right) \prod_{k=1}^{\infty}(1 - t^k).$$

Comparing this with the identity in the proof of Proposition 3.2, we obtain the result. □

Remark 3.5. When $q \to \infty$, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{(t;t)_n} = \sum_{\mathbf{p}\in\mathcal{P}} t^{|\mathbf{p}|} = \sum_{n=0}^{\infty} p(n) t^n.$$

This is a special case of [1, Corollary 2.2].

We generalize Proposition 3.2 to the case of multi-partitions. For a multi-partition $\mathbf{p} = (\rho^{(1)}, \dots, \rho^{(n)}) \in \mathcal{P}(n)$, we define

$$\kappa_q(\mathbf{p}) = \prod_{i=1}^{n} \kappa_q(\rho^{(i)}),$$

and for $k \geq 1$

$$\epsilon_{q,n}(k) = \sum_{\substack{\mathbf{p}\in\mathcal{P}(n) \\ |\mathbf{p}|=k}} \kappa_q(\mathbf{p}), \tag{3.6}$$

and set $\epsilon_{q,n}(0) = 1$. From the definitions, we have

$$\prod_{k=1}^{\infty}(1-q^{-1}t^k)^n = 1 + \sum_{\mathbf{p}\in\mathcal{P}(n)} \kappa_q(\mathbf{p}) t^{|\mathbf{p}|} = \sum_{k=0}^{\infty} \epsilon_{q,n}(k) t^k.$$

One can see that if $k > 0$, we have $\epsilon_{\infty,n}(k) = 0$.

Remark 3.7. Note that $\epsilon_{1,n}(k)$ is a classical arithmetic function related to modular forms. For example, we have $\epsilon_{1,24}(k) = \tau(k+1)$, where $\tau(k)$ is the Ramanujan τ-function. Thus the function $\epsilon_{q,n}(k)$ should be considered as a q-deformation of the function $\epsilon_{1,n}(k)$.

We also define for $k \geq 1$

$$p_{q,n}(k) = \sum_{\substack{\mathbf{p}\in\mathcal{P}(n) \\ |\mathbf{p}|=k}} (1-q^{-1})^{d(\mathbf{p})},$$

and set $p_{q,n}(0) = 1$. Notice that if $k > 0$, the function $p_{\infty,n}(k)$ is nothing but the multi-partition function with n-components. Hence we can think of $p_{q,n}(k)$ as a q-deformation of the multi-partition function.

Proposition 3.8. *If $k > 0$, then*

$$\epsilon_{q,n}(k) = \sum_{r=0}^{k} \epsilon_{1,n}(r) p_{q,n}(k-r). \tag{3.9}$$

Proof. We obtain from Proposition 1.9

$$\prod_{k=1}^{\infty}(1-q^{-1}\mathbf{z}^{k\delta})^n = \left(\sum_{\mathbf{p}\in\mathcal{P}(n)} (1-q^{-1})^{d(\mathbf{p})} \mathbf{z}^{|\mathbf{p}|\delta}\right) \prod_{k=1}^{\infty}(1-\mathbf{z}^{k\delta})^n.$$

After the change of variables $\mathbf{z}^\delta = t$, we obtain from the definitions

$$\begin{aligned}\sum_{k=0}^{\infty} \epsilon_{q,n}(k) t^k &= \left(\sum_{\mathbf{p} \in \mathcal{P}(n)} (1-q^{-1})^{d(\mathbf{p})} t^{|\mathbf{p}|} \right) \prod_{k=1}^{\infty} (1-t^k)^n \\ &= \left(\sum_{r=0}^{\infty} p_{q,n}(r) t^r \right) \left(\sum_{s=0}^{\infty} \epsilon_{1,n}(s) t^s \right).\end{aligned}$$

Now the identity (3.9) is clear. □

By taking $q \to \infty$, we obtain the following identity

$$0 = \sum_{r=0}^{k} \epsilon_{1,n}(r) p_{\infty,n}(k-r), \qquad k > 0,$$

where $p_{\infty,n}(k)$ is the multi-partition function with n-components. This classical identity is also a consequence of the following identities:

$$\prod_{k=1}^{\infty} (1-t^k)^n = \sum_{k=0}^{\infty} \epsilon_{1,n}(k) t^k \quad \text{and} \quad \prod_{k=1}^{\infty} (1-t^k)^{-n} = \sum_{k=0}^{\infty} p_{\infty,n}(k) t^k.$$

Example 3.10. When the affine Kac-Moody algebra $\mathfrak{g}$ is of type $X_{24}^{(1)}$, $X = A, B, C$ or D, we have

$$\epsilon_{q,24}(k) = \sum_{r=0}^{k} \tau(r+1) p_{q,24}(k-r)$$

and

$$0 = \sum_{r=0}^{k} \tau(r+1) p_{\infty,24}(k-r),$$

where $\tau(k)$ is the Ramanujan τ-function. If $k = 2$, the first identity becomes

$$\epsilon_{q,24}(2) = \tau(1) p_{q,24}(2) + \tau(2) p_{q,24}(1) + \tau(3) p_{q,24}(0).$$

Through some computations, we obtain

$$\epsilon_{q,24}(2) = 276 q^{-2} - 24 q^{-1}.$$

On the other hand, we have

$$\begin{aligned} & \tau(1)p_{q,24}(2)+\tau(2)p_{q,24}(1)+\tau(3)p_{q,24}(0) \\ = \ & p_{q,24}(2)-24p_{q,24}(1)+252 \\ = \ & \{276(1-q^{-1})^2+48(1-q^{-1})\}-24\cdot 24(1-q^{-1})+252 \\ = \ & 276(1-q^{-1})^2-528(1-q^{-1})+252 \\ = \ & 276q^{-2}-24q^{-1}=\epsilon_{q,24}(2). \end{aligned}$$

We also see that

$$\tau(1)p_{\infty,24}(2)+\tau(2)p_{\infty,24}(1)+\tau(3)p_{\infty,24}(0)=324-24\cdot 24+252=0.$$

Now we consider the whole set of positive roots, not just the set of imaginary positive roots, and obtain interesting identities. We begin with the identity (2.3). Recalling the description of the set of positive roots, we obtain

$$\begin{aligned} \sum_{\mu\in Q_+} H_\rho(\mu;q)\mathbf{z}^{-\mu} &= \mathbf{z}^{-\rho}\chi_q(V(\rho)) = \prod_{\alpha\in\Delta_+}(1-q^{-1}\mathbf{z}^{-\alpha})^{\operatorname{mult}\alpha} \\ &= \left(\prod_{k=1}^{\infty}(1-q^{-1}\mathbf{z}^{-k\delta})^n\prod_{\alpha\in\Delta_{\mathrm{cl}}}(1-q^{-1}\mathbf{z}^{\alpha-k\delta})\right)\prod_{\alpha\in\Delta_{\mathrm{cl}}^+}(1-q^{-1}\mathbf{z}^{-\alpha}), \end{aligned} \tag{3.11}$$

where Δ_{cl} is the set of classical roots.

Let $\mathcal{Z}=\{\sum_{\alpha\in Q_+}c_\alpha\mathbf{z}^{-\alpha}\,|\,c_\alpha\in\mathbb{C}\}$ be the set of (infinite) formal sums. Recall that we have the element $d\in\mathfrak{h}$ such that $\alpha_0(d)=1$ and $\alpha_j(d)=0$, $j\in I\setminus\{0\}$. Let $\mathfrak{h}_{\mathbb{Z}}$ be the $\mathbb{Z}$-span of $\{h_0,h_1,\dots,h_n,d\}$. We define the evaluation map $EV_t:\mathcal{Z}\times\mathfrak{h}_{\mathbb{Z}}\to\mathbb{C}[[t]]$ by

$$EV_t\left(\sum_\alpha c_\alpha\mathbf{z}^{-\alpha},\mathbf{s}\right)=\sum_\alpha c_\alpha t^{\alpha(\mathbf{s})},\qquad \mathbf{s}\in\mathfrak{h}_{\mathbb{Z}}.$$

Then we see that $EV_t(\cdot,d)$ is the same as the *basic specialization* in [14, p.219] with q replaced by t. We apply $EV_t(\cdot,d)$ to (3.11) and obtain

$$(1-q^{-1})^{|\Delta_{\mathrm{cl}}^+|}\prod_{k=1}^{\infty}(1-q^{-1}t^k)^{\dim\mathfrak{g}_{\mathrm{cl}}}=\sum_{k=0}^{\infty}\left(\sum_{\mu\in Q_{+,\mathrm{cl}}}H_\rho(k\alpha_0+\mu;q)\right)t^k, \tag{3.12}$$

where $\mathfrak{g}_{\mathrm{cl}}$ is the finite-dimensional simple Lie algebra corresponding to $\mathfrak{g}$, and $Q_{+,\mathrm{cl}}$ is the $\mathbb{Z}_{\geq 0}$-span of $\{\alpha_1,\dots,\alpha_n\}$. We write $|\Delta_{\mathrm{cl}}^+|=r$ and $\dim\mathfrak{g}_{\mathrm{cl}}=N$ so that $N=2r+n$. By comparing (3.12) with the identity $\prod_{k=1}^{\infty}(1-q^{-1}t^k)^n=\sum_{k=0}^{\infty}\epsilon_{q,n}(k)t^k$, we obtain:

Proposition 3.13.

$$\epsilon_{q,N}(k) = \sum_{\mu \in Q_{+,\mathrm{cl}}} H_\rho(k\alpha_0 + \mu; q) \Big/ (1 - q^{-1})^r.$$

By Definition 2.2, $\epsilon_{q,N}(k)$ is a power series in q^{-1} in the above formula. However, one can see from (3.6) that $\epsilon_{q,N}(k)$ is actually a polynomial in q^{-1}.

Example 3.14. We take $\mathfrak{g}$ to be of type $A_4^{(1)}$. Then the classical Lie algebra $\mathfrak{g}_{\mathrm{cl}}$ is of type A_4, and $r = |\Delta_{\mathrm{cl}}^+| = 10$ and $N = \dim \mathfrak{g}_{\mathrm{cl}} = 24$. Taking the limit $q \to 1$, we obtain

$$\tau(k+1) = \lim_{q \to 1} \sum_{\mu \in Q_{+,\mathrm{cl}}} H_\rho(k\alpha_0 + \mu; q) \Big/ (1 - q^{-1})^{10}.$$

Therefore the sum $\sum_{\mu \in Q_{+,\mathrm{cl}}} H_\rho(k\alpha_0 + \mu; q)$ is always divisible by $(1-q^{-1})^{10}$. However, the famous Lehmer's conjecture predicts that the sum is never divisible by $(1 - q^{-1})^{11}$.

3.2. **Kostant's function and the polynomial $H_\lambda(\mu; q)$.** In this subsection, let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra (finite type) or an untwisted affine Kac-Moody algebra (affine type).

Definition 3.15. We define the functions $K_q^\infty(\mu)$ and $K_q^1(\mu)$ by

$$\sum_{\mu \in Q_+} K_q^\infty(\mu) \mathbf{z}^\mu = \prod_{\alpha \in \Delta_+} \left(\frac{1 - q^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha} \right)^{\mathrm{mult}\,\alpha} = \sum_{b \in \mathbf{G}} (1 - q^{-1})^{d(\phi(b))} \mathbf{z}^{\mathrm{wt}(b)}$$

and

$$\sum_{\mu \in Q_+} K_q^1(\mu) \mathbf{z}^\mu = \prod_{\alpha \in \Delta_+} \left(1 - q^{-1}\mathbf{z}^\alpha\right)^{-\mathrm{mult}\,\alpha} = \sum_{b \in \mathbf{G}} q^{-|\phi(b)|} \mathbf{z}^{\mathrm{wt}(b)}.$$

We set $K_q^\infty(\mu) = K_q^1(\mu) = 0$ if $\mu \notin Q_+$.

Remark 3.16.

(1) Note that both $K_\infty^\infty(\mu)$ with $q = \infty$ and $K_1^1(\mu)$ with $q = 1$ are equal to the classical Kostant's partition function $K(\mu)$. Hence both of them can be considered as q-deformation of Kostant's function.

(2) The function $K_q^1(\mu)$ was introduced by Lusztig [19] for finite types. See also S. Kato's paper [15]. On the other hand, the function $K_q^\infty(\mu)$ for finite types can be found in the work of Guillemin and Rassart [13].

We obtain from the Casselman-Shalika formula (Proposition 2.6)

$$\begin{aligned}\mathbf{z}^{-\lambda}\chi(V(\lambda)) &= \sum_{\beta\in Q_+}(\dim V(\lambda)_{\lambda-\beta})\mathbf{z}^{-\beta}\\ &= \mathbf{z}^{-\lambda-\rho}\chi_q(V(\lambda+\rho))\prod_{\alpha\in\Delta_+}(1-q^{-1}z^{-\alpha})^{-\operatorname{mult}\alpha}\\ &= \left(\sum_{\mu\in Q_+}H_{\lambda+\rho}(\mu;q)\mathbf{z}^{-\mu}\right)\left(\sum_{\nu\in Q_+}K_q^1(\nu)\mathbf{z}^{-\nu}\right).\end{aligned}$$

Therefore, we have a q-deformation of the Kostant's multiplicity formula:

Proposition 3.17.

$$\dim V(\lambda)_{\lambda-\beta} = \sum_{\mu\in Q_+}H_{\lambda+\rho}(\mu;q)K_q^1(\beta-\mu).$$

In order to see that this is indeed a q-deformation of the Kostant's multiplicity formula, we need to determine the value of $H_{\lambda+\rho}(\mu;1)$.

Lemma 3.18. *We have*

$$H_{\lambda+\rho}(\mu;1) = \begin{cases}(-1)^{\ell(w)} & \text{if } w\circ\lambda = -\mu \text{ for some } w\in W,\\ 0 & \text{otherwise,}\end{cases}$$

where we define $w\circ\lambda = w(\lambda+\rho)-\lambda-\rho$ for $w\in W$ and $\lambda\in P_+$.

Note that such an element $w\in W$ is unique if it exists, so there is no ambiguity in the assertion.

Proof. From Definition 2.2, we obtain

$$\sum_{\mu\in Q_+}H_{\lambda+\rho}(\mu;1)\mathbf{z}^{\lambda+\rho-\mu} = \sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w(\lambda+\rho)}.$$

The condition $\lambda+\rho-\mu = w(\lambda+\rho)$ is equivalent to $w\circ\lambda = -\mu$. It completes the proof. □

Now we take $q=1$ in Proposition 3.17 and use Lemma 3.18 to obtain the classical Kostant's multiplicity formula

$$\dim V(\lambda)_{\lambda-\beta} = \sum_{w\in W}(-1)^{\ell(w)}K(w\circ\lambda+\beta).$$

Note that the sum is actually a finite sum. Indeed, we have $w\circ\lambda < 0$ for each $w\in W$ and $w\circ\lambda+\beta\geq 0$ only for finitely many $w\in W$ for fixed $\lambda\in P_+$ and $\beta\in Q_+$. For the same reason, the sum in (3.23) below is also a finite sum.

Remark 3.19. We have obtained in the previous section (Corollary 2.10)

$$H_{\lambda+\rho}(\mu;\infty) = \dim V(\lambda)_{\lambda-\mu}, \tag{3.20}$$

$$H_{\lambda+\rho}(\mu;-1) = \dim(V(\lambda)\otimes V(\rho))_{\lambda+\rho-\mu}. \tag{3.21}$$

When $\mathfrak{g}$ is of finite type, we defined $H_\lambda(\mu;q)$ in [16] as in Definition 2.2, and we can prove the analogous results.

We derive a formula for $H_{\lambda+\rho}(\mu;q)$ in the following proposition.

Proposition 3.22.

$$H_{\lambda+\rho}(\mu;q) = \sum_{w\in W}(-1)^{\ell(w)}K_q^\infty(w\circ\lambda+\mu). \tag{3.23}$$

Proof. We have from the definitions

$$\begin{aligned}\chi_q(V(\lambda+\rho)) &= \sum_{\mu\in Q_+} H_{\lambda+\rho}(\mu;q)\mathbf{z}^{\lambda+\rho-\mu}\\ &= \left(\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w(\lambda+\rho)}\right)\left(\sum_{\nu\in Q_+}K_q^\infty(\nu)\mathbf{z}^{-\nu}\right).\end{aligned}$$

The identity in the proposition follows from expanding the product and comparing the coefficients. □

If we take the limit $q\to\infty$ in (3.23), we have from (3.20)

$$\dim V(\lambda)_{\lambda-\mu} = \sum_{w\in W}(-1)^{\ell(w)}K(w\circ\lambda+\mu),$$

which is again the classical Kostant's multiplicity formula.

If we take $q=-1$ in (3.23), we obtain from (3.21)

$$\dim(V(\lambda)\otimes V(\rho))_{\lambda+\rho-\mu} = \sum_{w\in W}(-1)^{\ell(w)}K_{-1}^\infty(w\circ\lambda+\mu). \tag{3.24}$$

This is a generalization of the formula in Theorem 1 of [13] to the affine case.

Example 3.25. Assume that $\mathfrak{g}$ is of type $A_1^{(1)}$. We write $\mu = m\alpha_0+n\alpha_1 = (m,n)\in Q_+$ and set $\lambda=0$ in (3.23). Through standard computation, we obtain that

$$\{w\rho+\mu-\rho\,|\,w\in W\} = \left\{(m-\frac{k(k+1)}{2}, n-\frac{k(k-1)}{2})\,\Big|\,k\in\mathbb{Z}\right\}.$$

Thus we have

$$H_\rho(m,n;q)=\sum_{k\in\mathbb{Z}}(-1)^k K_q^\infty(m-\frac{k(k+1)}{2},n-\frac{k(k-1)}{2}).$$

By taking the limit $q\to\infty$, we obtain for $(m,n)\neq(0,0)$

$$0=\sum_{k\in\mathbb{Z}}(-1)^k K(m-\frac{k(k+1)}{2},n-\frac{k(k-1)}{2}).$$

In this case, $K(m,n)$ counts the number of vector partitions of (m,n) into parts of the forms (a,a), $(a-1,a)$ or $(a,a-1)$. Then one can see that we have obtained (3.9) on p.148 in [11].

We further investigate properties of the function $H_\lambda(\mu;q)$. From the definitions of $K_q^\infty(\mu)$ and $K_q^1(\mu)$, we have

$$\begin{aligned}\left(\sum_{\mu\in Q_+}K_q^\infty(\mu)\mathbf{z}^\mu\right)\left(\sum_{\nu\in Q_+}K_q^1(\nu)\mathbf{z}^\nu\right) &= \prod_{\alpha\in\Delta_+}\left(\frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha}\right)^{\operatorname{mult}\alpha}\prod_{\alpha\in\Delta_+}\left(1-q^{-1}\mathbf{z}^\alpha\right)^{-\operatorname{mult}\alpha}\\ &= \prod_{\alpha\in\Delta_+}(1-\mathbf{z}^\alpha)^{-\operatorname{mult}\alpha}=\sum_{\beta\in Q_+}K(\beta)\mathbf{z}^\beta,\end{aligned}$$

where $K(\beta)$ is the classical Kostant's function. Thus we have

$$\sum_{\mu\in Q_+}K_q^\infty(\mu)K_q^1(\beta-\mu)=K(\beta), \tag{3.26}$$

and we obtain, for $\beta>0$,

$$K_q^\infty(\beta)=K(\beta)-K_q^1(\beta)-\sum_{0<\nu<\beta}K_q^\infty(\nu)K_q^1(\beta-\nu), \tag{3.27}$$

and $K_q^\infty(0)=K_q^1(0)=K(0)=1$.

Then we obtain from Proposition 3.22

$$\begin{aligned}H_{\lambda+\rho}(\mu;q) &= H_{\lambda+\rho}(\mu;1)+\sum_{w\in W}(-1)^{\ell(w)}K(w\circ\lambda+\mu)-\sum_{w\in W}(-1)^{\ell(w)}K_q^1(w\circ\lambda+\mu)\\ &\qquad -\sum_{\substack{w\in W\\ w\circ\lambda+\mu>0}}(-1)^{\ell(w)}\sum_{0<\nu<w\circ\lambda+\mu}K_q^\infty(\nu)K_q^1(w\circ\lambda+\mu-\nu),\end{aligned}$$

where $H_{\lambda+\rho}(\mu;1)$ plays the role of correction term for the case $w\circ\lambda+\mu=0$. See Lemma 3.18 for the value of $H_{\lambda+\rho}(\mu;1)$. Also we used the fact that $K(\beta)=K_q^1(\beta)=K_q^\infty(\beta)=0$ unless $\beta\geq 0$.

Now we apply the classical Kostant formula and get the following proposition.

Proposition 3.28. *Assume that $\lambda \in P_+$ and $\mu \in Q_+$. Then we have*

$$\begin{aligned} H_{\lambda+\rho}(\mu;q) &= H_{\lambda+\rho}(\mu;1) + \dim V(\lambda)_{\lambda-\mu} - \sum_{w\in W} (-1)^{\ell(w)} K_q^1(w\circ\lambda+\mu) \\ &\quad - \sum_{\substack{w\in W \\ w\circ\lambda+\mu>0}} (-1)^{\ell(w)} \sum_{0<\nu<w\circ\lambda+\mu} K_q^\infty(\nu) K_q^1(w\circ\lambda+\mu-\nu). \end{aligned}$$

For the rest of this section, we assume that $\mathfrak{g}$ is of finite type. We denote by $\rho^\vee$ the element of $\mathfrak{h}$ defined by $\langle \alpha_i, \rho^\vee\rangle = 1$ for all the simple roots α_i. The following identity was conjectured by Lusztig [19] and proved by S. Kato [15].

Proposition 3.29. *For $\lambda \in P_+$ and $\mu \in Q_+$, we have*

$$\sum_{w\in W} (-1)^{\ell(w)} K_q^1(w\circ\lambda+\mu) = q^{-\langle \mu,\rho^\vee\rangle} P_{w_{\lambda-\mu}, w_\lambda}(q),$$

where w_ν is the element in the affine Weyl group $\hat{W}$ corresponding to $\nu \in P_+$ and $P_{w_{\lambda-\mu}, w_\lambda}(q)$ is the Kazhdan-Lusztig polynomial.

Hence we obtain from Proposition 3.28:

Corollary 3.30.

$$\begin{aligned} H_{\lambda+\rho}(\mu;q) &= H_{\lambda+\rho}(\mu;1) + \dim V(\lambda)_{\lambda-\mu} - q^{-\langle \mu,\rho^\vee\rangle} P_{w_{\lambda-\mu}, w_\lambda}(q) \\ &\quad - \sum_{\substack{w\in W \\ w\circ\lambda+\mu>0}} (-1)^{\ell(w)} \sum_{0<\nu<w\circ\lambda+\mu} K_q^\infty(\nu) K_q^1(w\circ\lambda+\mu-\nu). \end{aligned}$$

Setting $q = 1$, and noting that $K_1^\infty(\beta) = 0$ if $\beta > 0$, we see the famous property of the Kazhdan-Lusztig polynomial: $\dim V(\lambda)_{\lambda-\mu} = P_{w_{\lambda-\mu}, w_\lambda}(1)$.

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